

A model for the extracted heat and the phase front position in solidification with boundary condition of the third kind

G. C. J. BART and C. J. HOOGENDOORN

Delft University of Technology, Applied Physics Department, P.O. Box 5046, 2600 GA Delft, The Netherlands

(Received 13 August 1990)

Abstract—In this paper an approximate analytical solution is given for the extracted heat and the phase front position as a function of time for a finite slab of phase change material with a surface heat transfer coefficient and a fixed melting point. It will be shown how, for an arbitrary Biot number, an approximate solution can be obtained from the solutions in the case $Bi \rightarrow 0$ (homogeneous temperature in phase change material) and $Bi \rightarrow \infty$ (constant temperature boundary condition). The accuracy of the solution will be assessed by comparison with a numerical finite difference solution obtained by the enthalpy method.

1. INTRODUCTION

EXACT ANALYTICAL solutions are known for only a few phase change problems. One of the most useful and well-known exact solutions is the so-called Neumann solution of the Stefan problem. This is a solution for the melting or solidification of a semi-infinite slab with a constant temperature boundary condition. The phase change medium is characterized by a single transition point, equal density in both phases and other properties independent of temperature in the solid and liquid phases. Conduction is the only heat transfer mechanism that is taken into account. Besides this, approximate solutions for other geometries and boundary conditions have been found with the heat balance integral method, perturbation theory or a variational principle. These solution methods have been extensively applied in relatively simple cases with the phase change medium at transition temperature. A broad review of these methods can be found in both Viskanta [1] and Crank [2].

For the case of the solidification of a slab with an initially overheated liquid or the classical Stefan problem in a medium of finite extent, a solution based on the heat balance integral method and the time-dependent perturbation theory has been proposed by Charach and Zoglin [3]. Recently, Prud'homme and Hung Nguyen [4] gave singular perturbation solutions for slabs, cylinders and spheres with different boundary conditions. As they proposed the heat transfer in the liquid to be characterized by a constant heat transfer coefficient they only needed to solve the heat equation in the solid part of the medium. Bart *et al.* [5] proposed a solution of the classical Stefan problem in a finite slab by a combination of the solutions proposed by Neumann for the semi-infinite domain. Their results are in good accordance with the solution of Charach and Zoglin [3].

In this paper we will extend the method given by Bart *et al.* [5] to handle a convective boundary condition. It will be shown that accurate results can be obtained for the extracted heat as a function of time. The phase front position is predicted less accurately. No solution is given for the temperature field. An advantage of the method presented here is that it can easily be generalized to cope with phase change materials with a transition range as shown by Bart and van der Laag [6].

Application of the method can be found in the description of, for example, a latent heat store or the casting of metals.

2. DESCRIPTION OF THE PROBLEM

Between the coordinates $x = 0$ and L (see Fig. 1), we have a medium that is liquid for temperatures above T_f and solid if the temperature is below T_f . Further, it is supposed that each phase has its own constant properties. However, the density of the two phases is the same and we assume that no convection occurs in the liquid phase. The plane $x = L$ has an adiabatic boundary condition and the plane $x = 0$ a boundary condition of the third kind that can be described using a constant heat transfer coefficient α . The equation of thermal diffusion for the solid (s) and liquid (l) region has to be solved:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}. \quad (1)$$

Here, depending on the phase, T is chosen as T_s or T_l respectively. a , the thermal diffusivity of the respective range, is given by a_s or a_l . Furthermore, we have to account for the boundary conditions:

$$x = 0, \quad t > 0, \quad \alpha(T_w - T_\infty) = \lambda \partial T / \partial x \quad (2)$$

NOMENCLATURE

a	thermal diffusivity
Bi	Biot number
c_p	specific heat
Fo	Fourier number
k	root of transcendental equation
L	length
q	integration variable
Q_r	extracted heat
r	latent heat
Ste	Stefan number
t	time
T	temperature
V	volume
x	length coordinate
y	integration limit.

Greek symbols

α	heat transfer coefficient
γ	ratio of thermal diffusivities
η	ratio of sensible heats
λ	thermal conductivity
ξ	solidification front position
ρ	density
Φ	heat flux.

Subscripts

f	freezing
l	liquid
s	solid
w	wall
0	initial
∞	infinity.

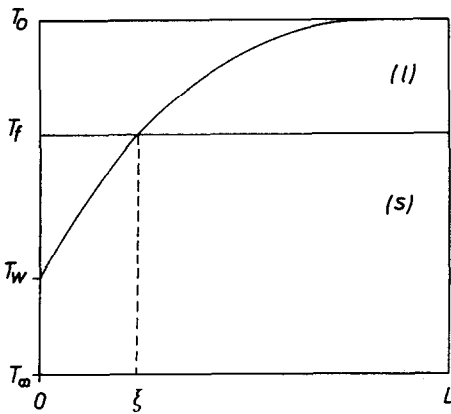


FIG. 1. Phase change material with transition point T_f at solidification front position ξ between the coordinates $x = 0$ and L . T_0 is the initial temperature of the liquid (l), T_∞ the temperature of the surrounding and T_w represents a wall temperature.

and

$$x = L, \quad t > 0, \quad \partial T / \partial x = 0. \quad (3)$$

Here also, depending on the phase, for wall temperature T_w , thermal conductivity λ and temperature T the values in the appropriate range should be taken.

The initial condition is

$$x > 0, \quad t = 0, \quad T_l = T_0. \quad (4)$$

At the phase front position ξ we have additional conditions given by the continuity of temperature

$$x = \xi, \quad t > 0, \quad T_s = T_l = T_f \quad (5)$$

and balance of heat fluxes

$$x = \xi, \quad t > 0, \quad \lambda_s \frac{\partial T_s}{\partial x} - \lambda_l \frac{\partial T_l}{\partial x} = r \rho \frac{\partial \xi}{\partial t}. \quad (6)$$

Here λ is the thermal conductivity, ρ the density and r the latent heat ($r > 0$). This set of equations is valid as long as not all the material has solidified. After solidification we only have to deal with the diffusion in the solid. This is described by energy equation (1) with boundary conditions as given by equations (2) and (3).

With a solution of equations (1) up to and including (6), the phase front position and temperature field as a function of time can easily be assessed. From this, other variables of interest can also be calculated, such as the heat flux at $x = 0$ or the extracted heat in a certain time interval.

An exact analytical solution of this general problem does not exist. However, for a few special cases exact or approximate analytical expressions can be obtained for the relative extracted heat Q_r and the phase front position ξ as a function of time. For application of the model in heat storage or casting problems, these are the most important quantities to know. In the next section we will give three different solutions for special cases. After that, with a combination of these solutions, we will give an approximate solution for the general problem.

The description of the problem can be simplified by introducing the Stefan number

$$Ste = c_{ps}(T_f - T_\infty)/r \quad (7)$$

and the sensible heat ratio

$$\eta = c_{pl}(T_0 - T_f)/c_{ps}(T_f - T_\infty). \quad (8)$$

Further, the Fourier number is defined as

$$Fo = a_s t / L^2 \quad (9)$$

the Biot number as

$$Bi = \alpha L / \lambda_s \quad (10)$$

and we will make use of the thermal diffusivity ratio

$$\gamma_{sl} = a_s/a_l. \quad (11)$$

3. SPECIAL ANALYTICAL SOLUTIONS

3.1. Solution with constant boundary temperature, $Bi \rightarrow \infty$

Carslaw and Jaeger (Section 11.2 of ref. [7]) found an exact solution for the case of a boundary condition of the first kind ($Bi \rightarrow \infty$, $T_w = T_\infty$), the phase change medium being semi-infinite. This solution is known as the Neumann solution of the Stefan problem. With the help of this analytical solution Bart *et al.* [5] proposed an approximate analytical solution for a medium of finite extent with constant temperature boundary condition.

In this approximate solution for short times the Neumann solution is followed. It is used up to the time when an amount of heat is extracted from the liquid phase equal to the initial heat present in the liquid in the finite domain. At that time the phase front position ξ is still within the region of x between 0 and L . After this time, denoted by the dimensionless Fourier number Fo_1 , we can use the Neumann solution for the case where the liquid is initially at the solidification temperature. With this change in solution regime the temperature field cannot be a continuous function of time, but with a time shift denoted by $Fo_2 - Fo_1$, the extracted heat becomes a continuous function of time. This second Neumann solution is used up to the time when the phase change is completed, or $\xi = L$. This time is the solidification time and will be denoted by Fo_3 . After this time we only have to deal with conduction in the solid. The solution is then taken from the well-known solution for the cooling of a slab with constant temperature boundary condition. In this case the temperature field is also discontinuous in time, but here again we can make the extracted heat a continuous function of time.

3.1.1. *Short time solution.* For short times we suppose the medium is semi-infinite and we can then follow the Neumann solution. To obey the boundary condition for $x = L$ the geometry is extended to $x = 2L$ with solidification taking place from both sides with the solution of the two semi-infinite regions added in the liquid region. This solution for the semi-infinite medium is valid as long as the penetration of the heat in the liquid has not yet reached the solidification front coming from the other side. Or better stated, the heat content in the tail of the temperature profile that has passed the solidification front is negligible. Then the position of the solidification front and the temperature profile in the solid will be the same as in the semi-infinite case.

According to the Neumann solution described by Carslaw and Jaeger [7], the thickness of the solidification layer is given by

$$\xi/L = 2k(Fo)^{1/2} \quad (12)$$

where k is the root of the equation

$$\frac{\exp(-k^2)}{k \operatorname{erf}(k)} - \frac{\eta}{\sqrt{\gamma_{sl}}} \frac{\exp(-k^2\gamma_{sl})}{k \operatorname{erfc}(k\sqrt{\gamma_{sl}})} = \frac{\sqrt{\pi}}{Ste}. \quad (13)$$

An evaluation of this equation is given by Churchill and Evans [8]. From the solution of the temperature field in the solid region

$$\frac{T_s - T_\infty}{T_f - T_\infty} = \frac{\operatorname{erf}(x/2L\sqrt{Fo})}{\operatorname{erf}(k)} \quad (14)$$

the heat flux at the wall ($x = 0$) turns out to be

$$\Phi'' = -\frac{\lambda_s(T_f - T_\infty)}{L(\pi Fo)^{1/2} \operatorname{erf}(k)}. \quad (15)$$

Integrating this equation with respect to time and normalizing the result gives an expression for the extracted relative energy

$$Q_r = \frac{2\sqrt{Fo}}{(1 + \eta + 1/Ste) \operatorname{erf}(k)\sqrt{\pi}}. \quad (16)$$

The calculated values for k for the cases simulated numerically can be found in Table 1. We will use this short time solution up to time Fo_1 , when according to the Neumann solution all sensible heat is extracted from the fluid and thus the fluid temperature in our finite domain equals the transition temperature. The Neumann solution gives, for the temperature of the fluid T_1 in the semi-infinite region

$$\frac{T_0 - T_1}{T_0 - T_f} = \frac{\operatorname{erfc}(x\sqrt{\gamma_{sl}}/L\sqrt{Fo})}{\operatorname{erfc}(k\sqrt{\gamma_{sl}})}. \quad (17)$$

The time Fo_1 is then calculated from

$$\int_{2k\sqrt{Fo_1}}^{\infty} (T_0 - T_1) dx/L = (1 - 2k\sqrt{Fo_1})(T_0 - T_f). \quad (18)$$

With use of the standard integral

$$\int_y^{\infty} \operatorname{erfc}(q) dq = \frac{\exp(-y^2)}{\sqrt{\pi}} - y \operatorname{erfc}(y) \quad (19)$$

we get the explicit expression from which Fo_1 can be obtained

$$\frac{\gamma_{sl}}{\sqrt{Fo_1}} \operatorname{erfc}(k\sqrt{\gamma_{sl}}) = \frac{2}{\sqrt{\pi}} \exp(-k^2\gamma_{sl}). \quad (20)$$

3.1.2. *Intermediate time solution.* For times $Fo \geq Fo_1$, the solution is obtained from the Neumann solution in case the sensible heat effect in the fluid can be neglected. Stating $\eta = 0$ and using k_0 instead of k in this case will turn equation (13) into

$$Ste \exp(-k_0^2) = \sqrt{\pi} k_0 \operatorname{erf}(k_0). \quad (21)$$

In the same way as described before this leads to a relative extracted heat

$$Q_r = \frac{1}{(1 + 1/Ste) \operatorname{erf}(k_0)} \frac{2}{\sqrt{\pi}} Fo^{1/2}. \quad (22)$$

Table 1. Parameters used in the simulations for the case $\eta = 1$, $\gamma_{st} = 1$

<i>Ste</i>	<i>k</i>	erf <i>k</i>	<i>k</i> ₀	erf <i>k</i> ₀	<i>Fo</i> ₁	<i>Fo</i> ₂	<i>Fo</i> ₃
0.1	0.18913	0.21090	0.22002	0.24431	0.52533	0.38828	5.30157
1.0	0.37776	0.40682	0.62006	0.61946	0.36763	0.14007	0.87779
10.0	0.46290	0.48730	1.25697	0.92454	0.31691	0.06185	0.41328

This solution will be valid as long as $\xi < L$. The calculated results for k_0 are also tabulated in Table 1. Equation (22) has to be modified somewhat, because now the initial condition at $Fo = Fo_1$ is given by the short time solution from Section 3.1.1. An extra term is added to account for the sensible heat effect of the liquid and along with it in the denominator the maximum extractable heat to which Q_r is related is changed. As we are solving the heat equation we prefer the Q_r vs time function to be continuous. That is why a time shift $Fo_2 - Fo_1$ is also needed. So from time $Fo = Fo_1$ onwards, the relative heat can be written as

$$Q_r(Fo \geq Fo_1) = \frac{2 \left(\frac{(Fo - Fo_1 + Fo_2)^{1/2}}{\text{erf}(k_0)} + \eta \right)}{(1 + \eta + 1/Ste)}. \quad (23)$$

Time Fo_2 is obtained from equations (16) and (23) by setting $Fo = Fo_1$. With this time shift the sum of the sensible heat extracted from the solid together with the latent heat extracted from the phase transition is the same in both the short time and intermediate time solution procedures at time Fo_1 . From Fo_1 onwards the solidification front position is given by

$$\xi/L = 2k_0(Fo - Fo_1 + Fo_2)^{1/2}. \quad (24)$$

Equation (23) can be used until time Fo_3 , where $\xi/L = 1$ or

$$2k_0\sqrt{(Fo_3 - Fo_1 + Fo_2)} = 1. \quad (25)$$

3.1.3. *Long time solution.* From time $Fo = Fo_3$ onwards, when all the liquid is solidified, the process is fully controlled by the diffusion in the solid and then the extracted heat can be approximated by the well-known long time solution:

$$\frac{1 - Q_r(Fo \geq Fo_3)}{1 - Q_r(Fo = Fo_3)} = \exp(-\pi^2(Fo - Fo_3)/4). \quad (26)$$

3.2. The solution for $Bi \rightarrow 0$

For the limit $Bi \rightarrow 0$ the heat flow rate will be completely determined by the external heat transfer coefficient. In this case the finite thermal conduction of the medium at each time will keep the temperature field within the material uniform and equal to the wall temperature. Balancing the heat flux at the wall with the change in heat content of the phase change material then yields

$$\alpha(T_w - T_\infty) = -\rho L c_p \frac{dT_w}{dt}. \quad (27)$$

Or in dimensionless form

$$-Bi c_{ps}/c_p dFo = d(T_w - T_\infty)/(T_w - T_\infty). \quad (28)$$

Here, for the specific heat c_p the appropriate value in the solid (c_{ps}) or liquid (c_{pl}) should be used. For $T_w > T_f$, the exact solution of this differential equation is

$$T_w - T_\infty = (T_0 - T_\infty) \exp(-Bi Fo c_{ps}/c_p). \quad (29)$$

This solution is valid up to time Fo_4 , where the liquid will reach the phase change temperature ($T_w = T_f$) with

$$Bi Fo_4 c_{ps}/c_{pl} = \ln((T_0 - T_\infty)/(T_f - T_\infty)). \quad (30)$$

After this time, during the phase change, the temperature of the phase change medium can be considered as constant and equal to the transition temperature. The extracted heat is the latent heat of the phase transition and is used to move the phase front position. In this case we have to solve

$$\alpha(T_f - T_\infty) = r\rho \frac{\partial \xi}{\partial t}. \quad (31)$$

Or in dimensionless form

$$Bi Ste = \frac{\partial(\xi/L)}{\partial Fo}. \quad (32)$$

Taking into account the initial condition that $\xi/L = 0$ at Fo_4 , we find as the solution of this equation

$$Bi Ste (Fo - Fo_4) = \xi/L. \quad (33)$$

This solution is valid up to time Fo_5 , when the phase front position reaches the adiabatic wall $\xi = L$, or

$$Fo_5 - Fo_4 = 1/Bi Ste. \quad (34)$$

After Fo_5 the exact solution for the temperature of the solid is given by

$$T_w - T_\infty = (T_f - T_\infty) \exp(-Bi(Fo - Fo_5)). \quad (35)$$

From the solution for the temperature the following equations for the relative extracted heat can easily be obtained:

for $0 \leq Fo \leq Fo_4$

$$Q_r = \frac{\eta(T_0 - T_w)/(T_0 - T_f)}{(1 + \eta + 1/Ste)}; \quad (36)$$

for $Fo_4 \leq Fo \leq Fo_5$

$$Q_r = \frac{\eta + (1/Ste)(Fo - Fo_4)/(Fo_5 - Fo_4)}{(1 + \eta + 1/Ste)}; \quad (37)$$

and for $Fo \geq Fo_5$

$$Q_r = \frac{\eta + 1/Ste + (T_f - T_{ws})/(T_f - T_\infty)}{(1 + \eta + 1/Ste)}. \quad (38)$$

3.3. The solution for $c_{pl} = c_{ps} = 0$ and arbitrary Bi

If we suppose the specific heat in the liquid and solid is negligible then the temperature profile in the solid will be a linear function with distance. The liquid temperature can then be set to T_f . Now the problem can be described by the following two equations: first the heat balance at the wall

$$\alpha(T_w - T_\infty) = \lambda_s(T_f - T_w)/\xi \quad (39)$$

and second the heat balance at the phase front position ξ

$$\lambda_s(T_f - T_w)/\xi = r\rho \frac{\partial \xi}{\partial t}. \quad (40)$$

In dimensionless form we obtain as a differential equation for ξ :

$$Bi Ste dFo = (1 + Bi \xi/L) d\xi/L. \quad (41)$$

After integration we obtain Fo as a function of the relative phase front position or the relative extracted heat, because in this case the extracted heat $Q_r = \xi/L$

$$Fo = Q_r/Bi Ste + Q_r^2/2Ste. \quad (42)$$

For $Bi \rightarrow 0$ we obtain

$$(Bi Fo)_{Bi \rightarrow 0} = Q_r/Ste \quad (43)$$

and for $Bi \rightarrow \infty$

$$(Fo)_{Bi \rightarrow \infty} = Q_r^2/2Ste. \quad (44)$$

We see that, in this special case, the solution for an arbitrary value of the Biot number for the relative extracted heat or the relative phase front position, as given in equation (42), can be obtained by an interpolation between the solution for $Bi \rightarrow 0$ and $Bi \rightarrow \infty$ according to

$$Fo = (Bi Fo)_{Bi \rightarrow 0}/Bi + (Fo)_{Bi \rightarrow \infty}. \quad (45)$$

4. THE GENERAL SOLUTION

4.1. Solution procedure

To get a solution for the general problem as stated in Section 2, with non-zero specific heats in the solid and liquid regions and an arbitrary heat transfer coefficient boundary condition, the result of equation (45), derived for a simplified case, will now be applied and combined with the results obtained for $Bi \rightarrow \infty$ (Section 3.1) and $Bi \rightarrow 0$ (Section 3.2).

If we want to know the time needed to extract an amount of heat Q_r in the case of an arbitrary Biot number, then we have to calculate from equation (16), (23) or (26) and the procedure given in Section 3.1 the value of $(Fo)_{Bi \rightarrow \infty}$ that corresponds to Q_r , and from equations (36) to (38) and the procedure given in Section 3.2 the value of $(Bi Fo)_{Bi \rightarrow 0}$ that corresponds to Q_r . Then, applying equation (45) will produce the Fo corresponding to Q_r in this general case.

If we want to know the dimensionless time Fo corresponding to the relative phase front position ξ/L in the case of an arbitrary Biot number then we first have to calculate the value of $(Fo)_{Bi \rightarrow \infty}$ from equation (12) or (24) and the value of $(Bi Fo)_{Bi \rightarrow 0}$ from equation (33). After application of equation (45) the Fourier number corresponding to ξ/L and for the arbitrary Biot number is found.

For a number of Biot values between 0.1 and 10 and for three different Stefan numbers 0.1, 1.0 and 10.0 we have done this and compared the results with those obtained with a finite difference solution of the problem. The finite difference numerical program could only cope with equal properties in the solid and liquid phases, so $\gamma_{sl} = 1$ in these simulations. In order to have a pronounced effect, if one exists, of the overheat, the sensible heat in the liquid has been chosen equal to that of the solid ($\eta = 1$). The finite difference numerical program is based on the enthalpy method as described by Voller and Cross [9]. In this method the phase front position is not tracked directly. It can be found afterwards by interpolation in the temperature field. Two hundred grid points have been used to obtain a smooth solidification front position curve.

The computing time of the FORTRAN code that has been used to calculate the approximate analytical solution is small. These calculations can be done on a microcomputer.

4.2. Results

The results for the extracted heat are presented in Fig. 2 and those for the phase front position in Fig. 3.

As explained in Section 3.1 the approximate model for the case $Bi \rightarrow \infty$ supposes the extracted heat to be a continuous function of time. This assumption is the reason why the phase front position shows a discontinuity for $Fo = Fo_1$. This discontinuity is most pronounced when k and k_0 differ considerably as is the case for large Stefan number and large initial overheat. As a result of our model we see that for the case $Bi \rightarrow \infty$ we have a good prediction of the relative extracted heat for all three Ste and for the phase front position that we have a good description for $Ste = 0.1$, a reasonable one for $Ste = 1.0$ and a rather poor one for $Ste = 10.0$. This is not a particular drawback of our model. To some extent all approximate models for Stefan problems as mentioned in the Introduction show such behaviour. For example, perturbation

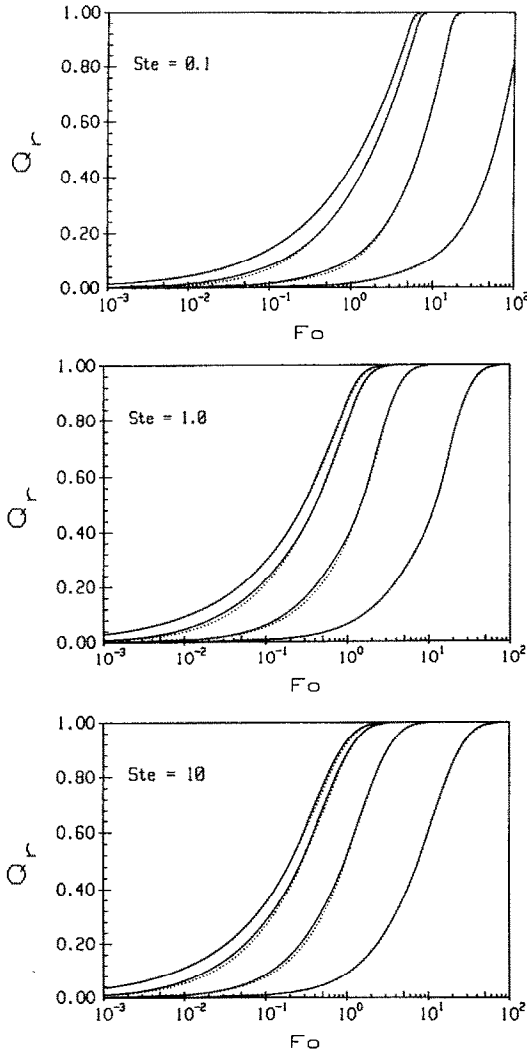


FIG. 2. Relative extracted heat Q_r as a function of the Fourier number Fo with the Biot number as parameter. Numerical calculations (.....) compared with the approximate analytical solution (—). From left to right the Biot number has been chosen as ∞ , 10, 1 and 0.1. The appropriate Ste is indicated on the graph.

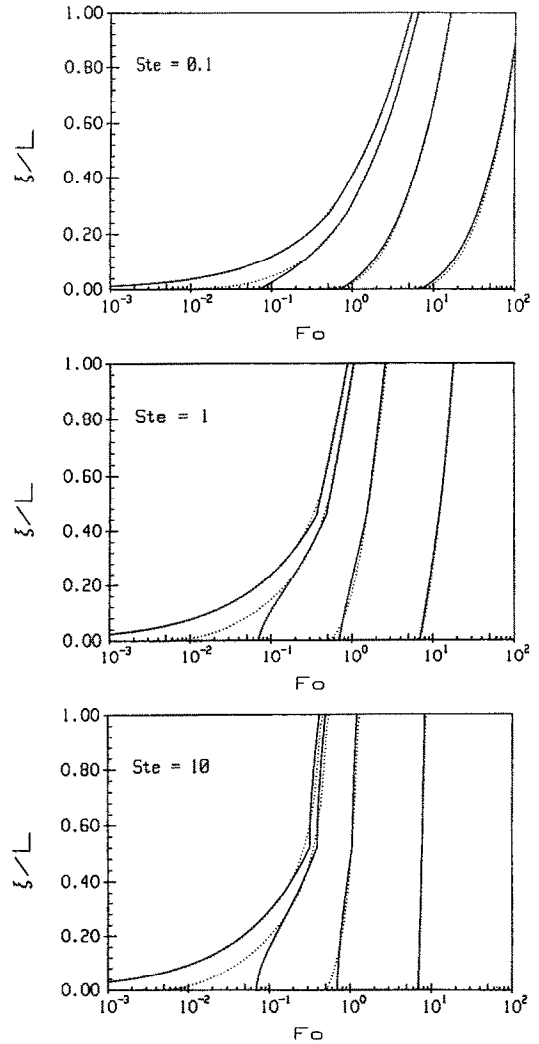


FIG. 3. Relative solidification front position ξ/L as a function of the Fourier number Fo with the Biot number as parameter. Numerical calculations (.....) compared with the approximate analytical solution (—). From left to right the Biot number has been chosen as ∞ , 10, 1 and 0.1. The appropriate Ste is indicated on the graph.

methods using Ste as the perturbation parameter will not converge when $Ste > 1$.

For the limit $Bi \rightarrow 0$ our model approaches the exact solution given in Section 3.2 for that case. The deviations that can be seen in the figures for $Bi = 0.1$ are predominantly due to the accuracy of the finite difference model. This model becomes less accurate with more and greater time steps used.

For intermediate Biot numbers we see deviations in the predicted relative extracted heat Q_r that are all smaller than 0.02. This is a value that is satisfactory, because it is comparable or better than the accuracy of the parameters of the model when these have to be measured.

In the prediction of the phase front position ξ/L , besides the effect of the discontinuity already discussed,

we see a considerable deviation for the initial movement of the phase front for $Bi = 10$ and to a lesser extent for $Bi = 1$. This effect becomes more severe for larger Ste . The existence of this discrepancy can also be noticed without the finite difference model (...), because the starting point of the curve is known exactly. When in the beginning there is only a temperature profile in the liquid, the time at which the wall temperature reaches T_f and solidification starts can be found from the exact solution that exists for the case of a constant heat transfer coefficient boundary condition and a medium without phase change. This is described in Section 3.10 of Carslaw and Jaeger [7]. In our case the time at which the solidification starts is found to be $Fo = 0.005915$ for $Bi = 10$, $Fo = 0.5120$ for $Bi = 1$ and $Fo = 6.821$ for $Bi = 0.1$. With some

interpolation between this starting point and the point of inflection that is visible in the curve that has been obtained with our approximate model it is possible to correct that curve. With this interpolation, use should be made of the fact that initially the ξ/L vs Fo curve shows a positive second derivative.

4.3. Conclusion

The separate model for the case $Bi \rightarrow \infty$ is determined by the primary parameters Ste , η and γ_{s1} . In the approximate description use is made of the secondary parameters k , k_0 , Fo_1 , Fo_2 and Fo_3 . The model gives a good description of the relative extracted heat for a wide range of Stefan numbers and for the solidification front position, a description that is accurate for $Ste \ll 1$ and less accurate for $Ste \gg 1$.

The model for the case $Bi \rightarrow 0$ is determined by the parameters Ste , η and c_{pl}/c_{ps} . With the calculation of $(BiFo_4)$ and $(BiFo_5)$ we have all the parameters necessary for the description of the extracted heat and solidification front position. This solution is an exact solution.

With the models for these two limiting cases and only one simple interpolation rule given by equation (45), a solution is obtained for an arbitrary Biot number. It has been shown that this solution for the solidification of a slab of phase change material with a boundary condition of the third kind yields quite reliable results. It gives a good description of the rela-

tive extracted heat as a function of time and useful information on the solidification front position.

REFERENCES

1. R. Viskanta, Phase-change heat transfer. In *Solar Heat Storage: Latent Heat Materials* (Edited by G. A. Lane), Chap. 5. CRC Press, Boca Raton, Florida (1983).
2. J. Crank, *Free and Moving Boundary Problems*. Clarendon Press, Oxford (1984).
3. Ch. Charach and P. Zoglin, Solidification in a finite, initially overheated slab, *Int. J. Heat Mass Transfer* **28**, 2261–2268 (1985).
4. M. Prud'homme et T. Hung Nguyen, Solutions par perturbations singulières pour un problème de Stefan généralisé, *Int. J. Heat Mass Transfer* **32**, 1501–1515 (1989).
5. G. C. J. Bart, C. J. Hoogendoorn and P. B. J. Schaareman, A characteristic dimensionless time in phase change problems, *Trans. ASME, J. Solar Energy Engng* **108**, 310–315 (1986).
6. G. C. J. Bart and P. C. van der Laag, Modelling of some constant heat transfer coefficient phase change problems. In *Heat Transfer 1990, Proc. Ninth Int. Heat Transfer Conf.*, Jerusalem, Israel (Edited by G. Hetsroni), Vol. 4, pp. 247–253. Hemisphere, New York (1990).
7. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd Edn), pp. 282–286. Clarendon Press, Oxford (1959).
8. S. W. Churchill and L. B. Evans, Coefficients for calculation of freezing in a semi-infinite region, *Trans. ASME, J. Heat Transfer* **93C**, 234–236 (1971).
9. V. R. Voller and M. Cross, Applications of control volume enthalpy methods in the solution of Stefan problems. In *Computational Techniques in Heat Transfer* (Edited by R. W. Lewis *et al.*), pp. 245–276. Pineridge Press, Swansea (1985).

UN MODELE POUR LA CHALEUR EXTRAITE ET LA POSITION DU FRONT DE PHASE DANS LA SOLIDIFICATION AVEC UNE CONDITION AUX LIMITES DE TROISIEME ESPECE

Résumé—On donne une solution analytique approchée pour la chaleur extraite et la position du front de phase en fonction du temps pour un matériau en plaque finie avec un coefficient de transfert thermique à la surface et un point de fusion fixe. On montre comment, pour un nombre de Biot arbitraire, une solution approchée peut être obtenue à partir du cas $Bi \rightarrow 0$ (température homogène dans le matériau à changement de phase) et $Bi \rightarrow \infty$ (condition limite de température constante). La précision de la solution peut être estimée en comparant avec une solution numérique aux différences finies obtenue par la méthode enthalpique.

EIN MODELL FÜR DIE ABGEFÜHRTE WÄRME UND DIE POSITION DER PHASENGRENZE BEI ERSTARRUNGSVORGÄNGEN MIT EINER RANDBEDINGUNG DER DRITTEN ART

Zusammenfassung—In der vorliegenden Arbeit wird eine analytische Näherungslösung für die zeitliche Entwicklung der abgeführten Wärme und der Position der Phasengrenze für den Erstarrungsvorgang in einer endlichen Platte angegeben. Dabei wird von einer Randbedingung der dritten Art an der Oberfläche und einem festen Schmelzpunkt ausgegangen. Es wird gezeigt, wie eine Lösung für eine beliebige Biot-Zahl näherungsweise aus Lösungen für $Bi \rightarrow 0$ (homogene Temperaturverteilung im Phasenwechselmaterial) und $Bi \rightarrow \infty$ (konstante Temperatur an der Oberfläche) bestimmt werden kann. Die Bewertung der Genauigkeit der Lösung erfolgt über einen Vergleich mit einer Lösung, die mittels Finite-Differenzen-Verfahren mit der Enthalpie-Methode numerisch berechnet wurde.

МОДЕЛЬ ОТВОДА ТЕПЛА И ПОЛОЖЕНИЯ ФРОНТА ФАЗОВОГО ПЕРЕХОДА В
ПРОЦЕССЕ ЗАТВЕРДЕВАНИЯ ПРИ ГРАНИЧНЫХ УСЛОВИЯХ 3-ГО РОДА

Аннотация—Дается приближенное аналитическое описание процесса отвода тепла и определение положения фронта фазового перехода в зависимости от времени для пластинных конечных размеров с известным коэффициентом теплопереноса на поверхности и постоянной точкой плавления. Показано, каким образом можно получить приближенное решение для произвольного числа Био из решений в случае $Bi \rightarrow 0$ (однородная температура материала при фазовом переходе) и $Bi \rightarrow \infty$ (граничное условие с постоянной температурой). Точность предложенного решения оценивается посредством сравнения с численным конечно-разностным решением, полученным методом эн-альпи.